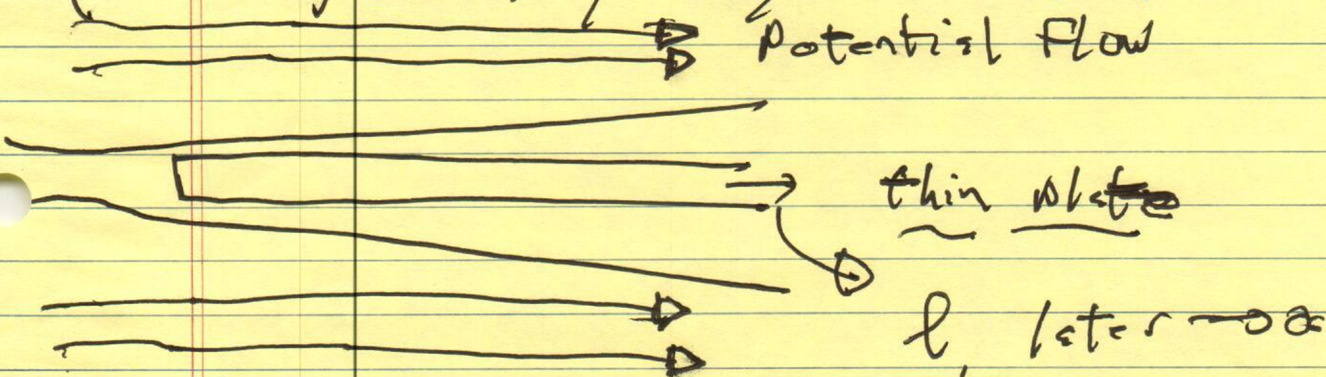
Lecture VIII – Laminar: Boundary Layers, Wakes

N.B.: For stability, see also:
P. Drazin, W.H. Reid: "Hydrodynamic Stability"

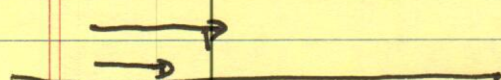
Here: – Laminar BL → $Re \gg 1$, but not turbulent
– Laminar Wake – see Lect. V.
(LL / Falkovich)

→ (Blasius) Boundary Layer Problem



Point: Boundary Layer

Point:



$v_n = 0|_p$

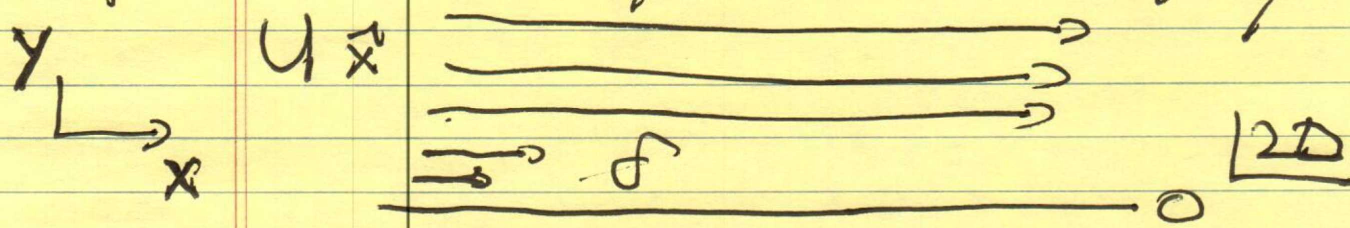
$v_t = 0|_p$

$Re \sim \frac{Ul}{\nu}$

– boundary layer forms in flow as consequence of higher Re

– layer is 'thin', as Re large.

– contrast Stokesian dynamics → viscosity dominant – everywhere so, for stationary flow in x, [redacted] y



$$\partial_x v_x + \partial_y v_y = 0$$

$$v_x \partial_x v_x + v_y \partial_y v_x = -\frac{1}{\rho}\partial_x P + \nu\left(\partial_x^2 v_x + \partial_y^2 v_x\right)$$

$$v_x \partial_x v_y + v_y \partial_y v_y = -\frac{1}{\rho}\partial_y P + \nu\left(\partial_x^2 v_y + \partial_y^2 v_y\right)$$

Now, further simplify:

→ $\underline{U}$ varies rapidly in y
  i.e. on order of the BL thickness

→ $\underline{\underline{so}}$ $|\partial_x^2 v_x| < |\partial_y^2 v_x|$
  i.e. along plate diff'n negligible

→ similarly, $|v_x| > |v_y|$
  and $|\partial_x P| \gg |\partial_y P|$

i.e.
  - streamlines flat
  - $|v_y| \ll |v_x|$ so

$\underline{\infty}$ $\quad p(x) + \frac{1}{2}\rho\, U(x)^2 = \text{const}.$

Pressure balance at Bernoulli, $\hat{x}$ direction

$\underline{\text{So}}$

$$\boxed{\begin{aligned} & v_x \partial_x v_x + v_y \partial_y v_x = -\frac{1}{\rho}\partial_x p + \nu \partial_y^2 v_x \\ & \partial_x v_x + \partial_y v_y = 0 \\ & \text{with: } \partial_x p = -\rho\, U \partial_x U \end{aligned}}$$

Prandtl equations.

with:

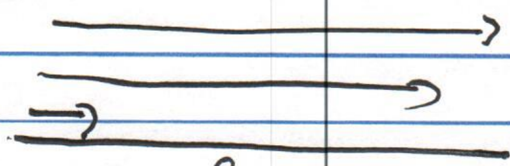
- b.c. $v_x = v_y = 0$ for $y = 0$
- $v_x = U(x)$ for $y \to \infty$

$\approx$

$\Rightarrow$

- can be generalized to other 2D flow.

Now



Proceed by BOE

- U_0 is flow as $y \to \infty$
- at BOE level:

$$\underbrace{v_x \partial_x v_x}_{\frac{U v_x}{[redacted]\, l}} + \underbrace{v_y \partial_y v_x}_{\text{(same)}} = \underbrace{-\frac{1}{\rho} \partial_x P}_{\text{ignore}} + \underbrace{\nu \partial_y^2 v_x}_{\frac{\nu}{\delta^2} v_x}$$

$$\underbrace{\partial_x v_x}_{} + \partial_y v_y = 0$$

$$\frac{v_x}{l\, [redacted]} \sim \frac{v_y}{\delta}$$

suggests variable change.

so:

$$\frac{U_0 v_x}{[redacted]\, l} \sim \frac{\nu}{\delta^2} v_x$$

$$\delta \sim \left(\nu [redacted] / U_0\right)^{1/2} \rightarrow \text{Blasius BL thickness.}$$

$$\rightarrow \underbrace{l}_{\text{plate scale.}} / Re^{1/2}$$

In case you are wondering:

$\hat{y}$: $\frac{P}{\rho\delta} \sim \frac{\nu V_y}{\delta^2}$ (if assume P not negligible)

$\frac{V_x}{l} \sim \frac{V_y}{\delta}$

$P \sim \rho \nu V_x / l$

$\Rightarrow$ $\frac{P}{\rho l} \sim \frac{\nu V_x}{l^2} \ll \frac{\nu V_x}{\delta^2}$

$\Rightarrow \frac{\nabla P}{\rho}$ neglible!

$$Re \sim U l / \nu$$

suggests:

$$x = l x' , \quad y = l y' / \sqrt{Re} ,$$

$$v_x = U_0 v_x' , \quad v_y = U_0 l v_y' / \sqrt{Re}$$

$$U = U_0 U$$

so, rescaled Prandtl eqns: (dim-less)

$$v_x' \frac{\partial}{\partial x'} v_x' + v_y' \frac{\partial v_x'}{\partial y'} - \frac{\partial^2 v_x'}{\partial y'^2} = U \frac{dU}{dx}$$

$$\frac{\partial v_x'}{\partial x'} + \frac{\partial v_y'}{\partial y'} = 0$$

Note: - Re scaled out
- change in Re ⇒ rescaling
- x unchanged
y $1/\sqrt{Re}$
Reynolds # similarity

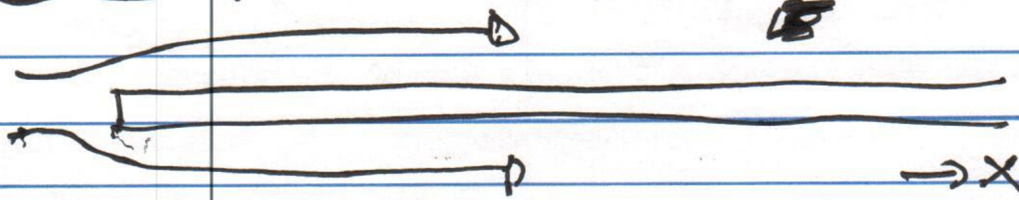
thus, immediately:

$$\delta \sim l / \sqrt{Re}$$

$$v_y \sim U / \sqrt{Re}.$$

Now, can specify to Blasius problem ⇒ semi-infinite plate.

Semi-infinite plate is rigorously solvable by similarity methods.



- here: expect δ thickens with x, as characteristic scale gone.

<u>i.e.</u> $\delta^2 \sim$ [redacted] νt

$\sim \nu x/U$

$\Rightarrow \delta \sim (\nu x/U)^{1/2}$.

$\Updownarrow$

Blasius B.L. thickness.

- $U = U(x) \to$ const. U_0

<u>so</u>

$$v_x \partial_x v_x + v_y \partial_y v_y = \nu \partial_y^2 v_x + \cancel{\frac{d}{dx} \rho U^2}$$

$$\partial_x v_x + \partial_y v_y = 0$$

Now, <u>no</u> characteristic length

<u>so</u>

- where before
$$x' = x/\ell$$
$$y' = y\,(U/\ell\nu)^{1/2}$$

Ref. on similarity methods : G.I. Barenblatt, "Scaling"

- Now, need get ℓ out of problem

so

$$- \quad v_x/U = F(y'/\sqrt{x'}) \qquad \frac{y'}{\sqrt{x'}} = y\sqrt{U/\nu x}$$

$v(x,y) \to F(y'/\sqrt{x'})$ → simplest possible not involving ℓ

→ self-sim.

$$- \quad v_y' \sqrt{x'} = G(y'/\sqrt{x'})$$

so $\quad y'/\sqrt{x'} \sim \varepsilon \to$ $\boxed{\text{self-similarity variable}}$

$$\Rightarrow \boxed{\delta \sim (\nu x/U)^{1/2}} \qquad \text{BL thickness}$$

To exploit:

$$\underline{v} = \left(\frac{\partial\psi}{\partial y}, \frac{-\partial\psi}{\partial x}\right) \qquad \text{cl: } \underline{O}\psi x \vec{z}$$

↳ to make $v_x \to U$

$$\psi = (\nu x U)^{1/2} f(\varepsilon)$$

self-similarity variable

$$\varepsilon = y'/\sqrt{x'} \to y(U/\nu x)^{1/2}$$

again : $\boxed{\delta \sim (\nu x/U)^{1/2}}$

Now

$$v_x \frac{\partial}{\partial x} v_x + v_y \frac{\partial}{\partial y} v_y = \nu\, \partial^2 v_x / \partial y^2$$

$$\psi = (\nu x U)^{1/2} f(\varepsilon)$$

$$\varepsilon = y\,(U/\nu x)^{1/2}$$

$$v_x = \partial\psi/\partial y$$

$$v_y = -\partial\psi/\partial x$$

$\Rightarrow$ Plug in :

$$\boxed{f f'' + 2 f''' = 0}$$

{ Num. — similarity variable reduces PDE → ODE Δ

$f(0) = f'(0) = 0 \,,\quad f'(\infty) = 1$ B.C.

and can solve { numerics; some asymptotics } see L^2

$$v_x = U f'(\varepsilon)$$

$$v_y = \tfrac{1}{2}\sqrt{\nu U/x}\,(\varepsilon f' - f)$$

⇒ More useful to estimate:

$$\sigma_{xy} = \eta \frac{\partial v_x}{\partial y}\Big|_0 \sim \eta \frac{U}{\delta}$$

↓
frictional force/area (stress) on plate

$$\sim \eta \left(U^3/\nu x\right)^{1/2} \underbrace{f''(0)}$$

$$\to .332 \left(\eta \rho U^3/x\right)^{1/2}$$

If plate has length ℓ,

$$\frac{F}{L} = \int_0^\ell dx\, \eta \frac{\partial v_x}{\partial y}\Big|_0 \sim \int_0^L dx\, \frac{\eta U}{(\nu x/U)^{1/2}}$$

↓
force/length

$$\sim \left(\eta \rho \ell U^3\right)^{1/2}$$

$$\sim U^{3/2} \ell^{1/2} \eta^{1/2} \Rightarrow F \sim U^{3/2} \ell^{3/2} \eta^{1/2}$$

Contrast stokes:
$6\pi \eta \ell U$.

contrast →

$$F \sim U \ell \eta$$

and, define drag coefficient C_d via:

defn.

$$C_d = \left[\frac{F}{L} \Big/ \tfrac{1}{2}\rho U^2 \right] 2\ell$$

2D defn.

: for 2D flow.

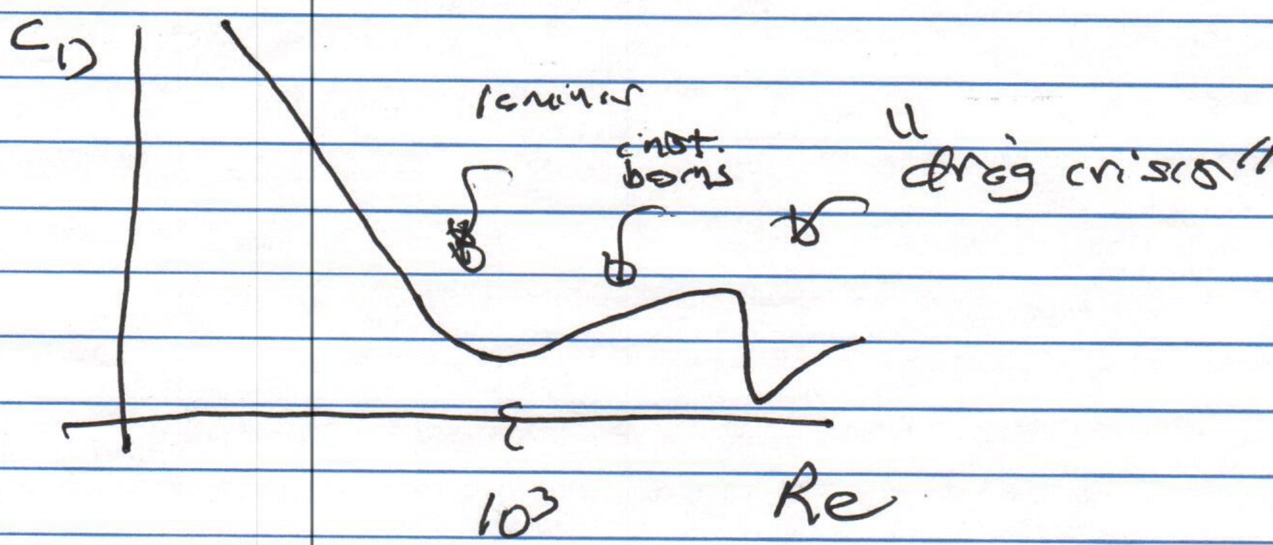
$$\sim \# \Big/ \sqrt{Re}$$

in general

$$C_d = F \Big/ \tfrac{1}{2}\rho U^2 S$$

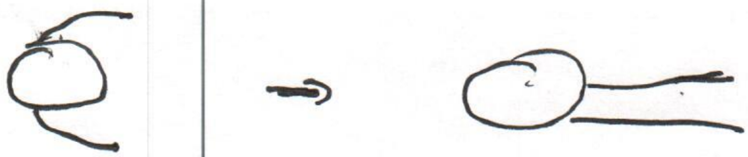
surface area.

Comments on Drag:



→ $Re > 1$ ⇒ laminar BL

→ $Re > 10^3$ ⇒ instability, separation begin, reach body. C_D rises



⇒ turbulence onset ⇒ C_D drops
'drag crisis'

~ B.L. energized

B.L. flow with turb. can beat viscous losses.

Why golf balls have dimples

~ C_D indep. Re ⇒ turbulence physics

→ Wakes

For laminar wake — see Lecture 7.

Point of wake: Viscosity changes global structure of flow.

C.f.: "Finnegans Wake"

by James Joyce.

⇒ classical Joyce novel
→ incomprehensible
→ origin of "quark" (word):

'3 quarks for Muster Mark'

⇒ not related to fluid dynamics except possibly beer consumption.